NECTO

An abstract game for 2 players by Kashio Fujii

Components

- Square tiles of 8 kinds (2 curves, 2 crosses, 4 twists). Every tile has a line end at the middle of each side, and two strands join them in pairs. A cross tile holds one crossing and a twist tile two.
- A square board of 6 × 6 cells (4 × 4 and 8 × 8 also work). A frame around the board joins the line ends on its edge, two neighboring ends at a time.



The eight tiles. From the left: curves c1 and c2, crosses x1 (vertical strand on top) and x2 (horizontal strand on top), and twists t1, t2, t3 and t4. The codes are the names used in the demo's game records.

Definitions

Loop A line that runs through tiles and the frame and comes back to where it started. When the board is full, every line is a loop.

Knot A loop that cannot be untangled into a circle without crossings, however it is moved, as long as each crossing keeps which strand is on top.

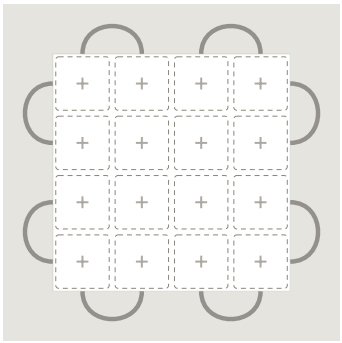
Named knots and their values The knots that can be drawn with at most 7 crossings, plus two trefoils (granny or square knot) and a trefoil with a figure-eight knot. A named knot is worth the fewest crossings any drawing of it needs (trefoil 3, figure-eight knot 4, knots of 5 crossings 5, … knots of 7 crossings 7, two trefoils 6, trefoil with figure-eight 7). More complex knots have no name and score nothing.

Identification The computer decides whether a loop is knotted and which knot it is (from its Alexander polynomial; on loops with very many crossings it may, rarely, mistake the type).

A loop's crossings The crossings where the loop crosses itself (crossings with other loops do not count). Each crossing belongs to the player who placed (or replaced) its tile.

Setup

- The board starts empty.
- Decide who plays first.



A 4 × 4 board and its frame. The frame joins the line ends on the edge of the board, two neighboring ends at a time.

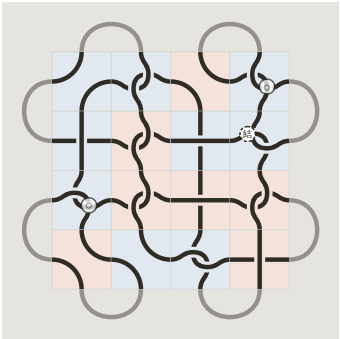
End of the Game

- Once the board is full, the game ends when the player to move has no tile to replace (every tile on the board is a curve).
- The player with the higher total wins. Equal totals are a draw.

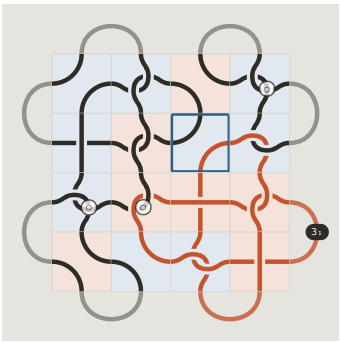
On Your Turn

The game has two stages: **placing**, until the board is full, and then **replacing**. Turns alternate throughout.

1. **Placing** (while the board has empty cells)
   Choose any empty cell and place any one of the 8 tiles on it. Nobody scores in this stage.
2. **Replacing** (once the board is full)
   - Choose a tile on the board and replace it with a tile that has **fewer crossings** (twist → cross or curve, cross → curve; any orientation). The new tile belongs to the player who placed it.
   - When a loop through the replaced cell **becomes a named knot** (one it was not already before the replacement), its value goes to **the player who placed more of that loop's crossings**. If both placed the same number, both players score it.
   - Points stay even if the loop changes later.



The replacing stage on a 4 × 4 board. Each tile is tinted by the player who placed it (first: blue, second: vermilion). The dashed label marks a knot too complex to have a name.



First replaces the cross on C3 with a curve, and one loop becomes a trefoil (3_1). That loop's crossings are first 2, second 4, so the 3 points go to Second (the loop is drawn in vermilion). Points go to whoever placed more of the loop's crossings, not to whoever made the replacement.

The Mathematics of Knots

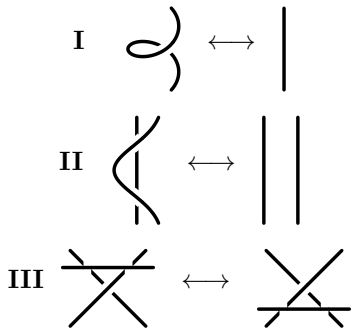
NECTO's scoring is based on ideas from knot theory, a branch of mathematics. This section explains what you need to understand the game.

Knot diagrams

When the board is full, every line is a loop. A loop is a closed curve drawn in the plane, and wherever two strands meet, the tile decides which one passes on top. In mathematics, a loop drawn in the plane with this over/under information is called a **knot diagram**. In a diagram, the strand passing underneath is drawn with a break.

When are two knots the same?

Stretching or moving a string (without cutting and rejoining it) does not change the knot itself. Its diagram, however, can change in endless ways. Three local changes of a diagram, the **Reidemeister moves**, are the following.



Move I: add or remove a twist. Move II: slide one strand over another, or pull them apart (two crossings appear or disappear). Move III: slide a strand over (or under) a crossing.

Two diagrams show the same knot exactly when one can be turned into the other by repeating these three moves (together with deformations in the plane that keep the crossings unchanged). This is Reidemeister's theorem.

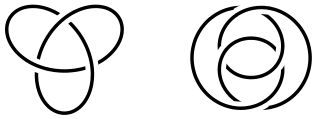
Unknots and knots

A loop that can be turned into a circle without crossings is called the **unknot**. Any other loop is what the game calls a **knot**.

A loop with many crossings is often still an unknot. Twists like the one on the left of move I, for example, can be removed however many there are. In fact, every diagram with two crossings or fewer is an unknot: a loop needs at least three crossings with itself to be knotted.

Crossing number

One knot can be drawn by many diagrams with different numbers of crossings. The smallest of these numbers is the knot's **crossing number**. In NECTO, a knot is worth its crossing number.



The trefoil 3_1 (crossing number 3) and the figure-eight knot 4_1 (crossing number 4).

A loop on the board usually has far more crossings than its crossing number. What scores is not the number of crossings in the diagram, but the number that no redrawing of the loop can get rid of.

Names of knots

Knots are named by their crossing number and an index: 3_1 means "the first knot with crossing number 3". Up to crossing number 7 there are 14 knots (counting a knot and its mirror image as the same):

Name	Points	Alexander polynomial
3_1	3	$t^2 - t + 1$
4_1	4	$t^2 - 3t + 1$
5_1	5	$t^4 - t^3 + t^2 - t + 1$
5_2	5	$2t^2 - 3t + 2$
6_1	6	$2t^2 - 5t + 2$
6_2	6	$t^4 - 3t^3 + 3t^2 - 3t + 1$
6_3	6	$t^4 - 3t^3 + 5t^2 - 3t + 1$
7_1	7	$t^6 - t^5 + t^4 - t^3 + t^2 - t + 1$
7_2	7	$3t^2 - 5t + 3$
7_3	7	$2t^4 - 3t^3 + 3t^2 - 3t + 2$
7_4	7	$4t^2 - 7t + 4$
7_5	7	$2t^4 - 4t^3 + 5t^2 - 4t + 2$
7_6	7	$t^4 - 5t^3 + 7t^2 - 5t + 1$
7_7	7	$t^4 - 5t^3 + 9t^2 - 5t + 1$
$3_1\#3_1$	6	$t^4 - 2t^3 + 3t^2 - 2t + 1$
$3_1\#4_1$	7	$t^4 - 4t^3 + 5t^2 - 4t + 1$

The last two are two knots tied one after the other in a single string, called a **connected sum** (written #). Two trefoils can be joined as the granny knot (both trefoils of the same handedness) or the square knot (mirror images of each other); NECTO counts both as $3_1\#3_1$.

How the computer decides

Telling a knot from an unknot just by looking at a diagram is not easy. So NECTO uses the **Alexander polynomial**, which can be calculated from a diagram. Reidemeister moves do not change it, so if two polynomials differ, the knots differ.

The polynomials in the table are all different, so the polynomial tells these types apart. (A polynomial is fixed only up to multiplying by a power of t and changing its overall sign; the table writes each with a positive constant term.) The converse does not hold: different knots can share a polynomial. The Kinoshita–Terasaka knot, with crossing number 11, has the polynomial 1, the same as the unknot. This is why the type may "rarely be mistaken" on loops with very many crossings.

The Alexander polynomial also cannot tell a knot from its mirror image, so in NECTO the right-handed and left-handed trefoils are the same 3_1.

What a replacement does

Replacing a cross tile with a curve tile removes that crossing and changes how the strands connect. If the two strands at the crossing belong to the same loop, the loop either splits into two or stays one, depending on how the curve is turned. If they belonged to two different loops, the two loops become one. So every replacement can change the shapes and the number of loops, and with them the types of the knots.

Every replacement lowers the total number of crossings on the board by at least one. The replacing stage therefore always ends within as many turns as the board had crossings when it began. At the end there are no crossings left, so every loop is an untangled circle.